Classification of arbitrary-dimensional multipartite pure states under stochastic local operations and classical communication using the rank of coefficient matrix

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We study multipartite entanglement under stochastic local operations and classical communication (SLOCC) and propose the entanglement classification under SLOCC for arbitrary-dimensional multipartite (n-qudit) pure states via the rank of coefficient matrix, together with the permutation of qudits. The entanglement classification of the $2 \otimes 2 \otimes 2 \otimes 4$ system is discussed in terms of the generalized method, and 19 different SLOCC families are found.

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Entanglement plays a vital role in quantum information processing, which includes quantum teleportation, quantum cryptography, quantum computation, etc [1]. Classification of different types of multipartite entanglement has been one of the main tasks in quantum information theory. Many studies on multipartite entanglement classification under different restrictions, such as local operations and classical communication (LOCC) and stochastic LOCC (SLOCC) [2, 3], have been conducted in recent years. It has been shown that if two pure states $|\psi\rangle$ and $|\phi\rangle$ are connected by SLOCC, they can be converted into each other with the product of invertible local operators (ILOs)

$$|\phi\rangle = A \otimes B \otimes C \, |\psi\rangle \, , \tag{1}$$

and are therefore called SLOCC equivalent. Two pure states that are equivalent under SLOCC can perform the same quantum information tasks [4].

The main idea of entanglement classification is to find an invariant preserved under SLOCC, and considerable research has been conducted on the entanglement classification of three [4], four [5–10] and n-qubit pure states [11–14] under SLOCC since the beginning of this century. Recently, Li *et al.* have proposed a simpler and more efficient approach for SLOCC classification of general n-qubit pure states in Ref. [15]. A general n-qubit pure state can be expanded as $|\psi\rangle = \sum_{i=0}^{2^n-1} a_i \, |i\rangle$, where a_i are the coefficients and $|i\rangle$ are the binary basis states. The coefficient matrix is constructed as follows:

$$M(|\psi\rangle) = \begin{pmatrix} a_{\underbrace{0\cdots0}_{[n/2]}\underbrace{0\cdots0}_{[(n+1)/2]}} & \cdots & a_{\underbrace{0\cdots0}_{[n/2]}\underbrace{1\cdots1}_{[(n+1)/2]}} \\ a_{\underbrace{0\cdots1}_{[n/2]}\underbrace{0\cdots0}_{[(n+1)/2]}} & \cdots & a_{\underbrace{0\cdots1}_{[n/2]}\underbrace{1\cdots1}_{[(n+1)/2]}} \\ \vdots & \vdots & \vdots \\ a_{\underbrace{1\cdots1}_{[n/2]}\underbrace{0\cdots0}_{[(n+1)/2]}} & \cdots & a_{\underbrace{1\cdots1}_{[n/2]}\underbrace{1\cdots1}_{[(n+1)/2]}} \end{pmatrix} \tag{2}$$

where the subscripts of the coefficients are written in binary form. For two n-qubit pure states connected by SLOCC, Li *et al.* proved that the rank of the coefficient matrices are equal whether or not the permutation of qubits is fulfilled on both states. This theorem provides a way of partitioning all the n-qubit states into different families.

With the development of quantum information theory, the importance of qudit is gradually recognized. Maximally entangled qudits have been shown to violate local realism more strongly and are less affected by noise than qubits [16–18]. Using Entangled qudits can provide more secure scheme against eavesdropping attacks in quantum cryptography [19–23], and also offers advantages including greater channel capacity for quantum communication [24] as well as more reliable quantum processing [25]. Much efforts has been put on the classification of bipartite and tripartite states with higher dimensions in systems such as $2 \otimes 2 \otimes n$ [26, 27], $2 \otimes n \otimes n$ [28], $2 \otimes m \otimes n$ [29–31] and $m \otimes n \otimes n$ [32].

In this paper, we generalize the concept of coefficient matrix to n-qudit pure states. A theorem is provided to show that the rank of the coefficient matrix is invariant under SLOCC. By calculating the rank of coefficient matrix along with the permutation of qudits, we successfully obtain the results of classification for n-qudit pure states under SLOCC. We investigate several examples and interesting entanglement properties are discovered. Using our theorem, we discuss the entanglement classification of the $2 \otimes 2 \otimes 2 \otimes 4$ system, which we believe has never been studied before.

Suppose an n-qudit pure state $|\psi\rangle$ in the n-partite Hilbert space $\mathcal{H}^n = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ have the dimensions $d_1, d_2 \cdots, d_n$, respectively, which can be expanded in the form

$$|\psi\rangle = \sum_{i=0}^{\prod_{k=1}^{n} d_k - 1} a_i \, |s_1 s_2 \cdots s_n\rangle, \tag{3}$$

where a_i are the coefficients and $|s_1 s_2 \cdots s_n\rangle$ are the basis states

$$|s_1 s_2 \cdots s_n\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_n\rangle \tag{4}$$

with $s_k \in \{0, 1, \cdots, d_k - 1\}, k = 1, \cdots, n$. The coefficient matrix $M(|\psi\rangle)$ is constructed by arranging $a_i (i = 0, \cdots, \prod_{k=1}^{n} d_k -$

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1) in lexicographical ascending order

$$M(|\psi\rangle)=\begin{pmatrix} a_{\underbrace{0\cdots0}_{l}\underbrace{0\cdots0}_{n-l}} & \cdots & a_{\underbrace{0\cdots0}_{l}\underbrace{d_{n-l}-1\cdots d_n-1}_{n-l}} \\ a_{\underbrace{0\cdots1}_{l}\underbrace{0\cdots0}_{n-l}} & \cdots & a_{\underbrace{0\cdots1}_{l}\underbrace{d_{n-l}-1\cdots d_n-1}_{n-l}} \\ \vdots & \vdots & \vdots \\ a_{\underbrace{d_1-1\cdots d_l-1}_{l}\underbrace{0\cdots0}_{n-l}} & \cdots & a_{\underbrace{d_1-1\cdots d_l-1}_{l}\underbrace{d_{n-l}-1\cdots d_n-1}_{n-l}} \end{pmatrix}_{\prod_{k=1}^{l}d_k\times\prod_{k=l+1}^{n}d_k} \tag{5}$$

where $1 \le l \le n-1$.

All permutations of qudits are included in the set

$$\{\sigma\}=\{(r_1,c_1)(r_2,c_2)\cdots(r_k,c_k)\} \tag{6}$$

where $1 \le r_1 < r_2 < \cdots < r_k < l+(n \mod 2)$, $l < c_1 < c_2 < \cdots < c_k \le n$, and (r_i, c_i) represents the transposition of r_i and c_i. The purpose of choosing the permutation form in Eq. (6) is to omit the permutations that end up exchanging rows or columns in the coefficient matrix. Letting k vary from 0 to $l-(n \mod 2)$, and we get all the elements included in the set $\{\sigma\}$. The case where $k=0$ is defined as identical permutation, denoted by $\sigma = I$. Each element σ of the set $\{\sigma\}$ gives a permutation $\{q_1, q_2, \cdots, q_n\}$ of $\{1, 2, \cdots, n\}$.

Theorem. Suppose that two n-qudit pure states are in the n-partite Hilbert space $\mathcal{H}^n = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ have the dimensions $d_1, d_2 \cdots, d_n$, respectively. If they are SLOCC equivalent, then they can be expressed as

$$|\psi\rangle = F_{(1)} \otimes F_{(2)} \otimes \cdots \otimes F_{(n)} |\phi\rangle \tag{7}$$

where $F_{(1)}, F_{(2)}, \cdots, F_{(n)}$ are ILOs in $GL(d_1, \mathbb{C})$, $GL(d_2, \mathbb{C})$, $\cdots$, $GL(d_n, \mathbb{C})$, respectively.

The coefficient matrices of $|\psi\rangle$ and $|\phi\rangle$ satisfy the relation

$$\begin{aligned} M(|\psi\rangle) &= (F_{(1)} \otimes \cdots \otimes F_{([n/2])}) \\ &\quad \times M(|\phi\rangle)(F_{([n/2]+1)} \otimes \cdots \otimes F_{(n)})^T. \end{aligned} \tag{8}$$

Applying permutation σ to both sides of Eq. (8) gives

$$\begin{aligned} M^\sigma(|\psi\rangle) &= (F^\sigma_{(1)} \otimes \cdots \otimes F^\sigma_{([n/2])}) \\ &\quad \times M^\sigma(|\phi\rangle)(F^\sigma_{([n/2]+1)} \otimes \cdots \otimes F^\sigma_{(n)})^T, \end{aligned} \tag{9}$$

which indicates that $M^\sigma(|\psi\rangle)$ and $M^\sigma(|\phi\rangle)$ have the same rank. The detailed proof is given in appendix.

Therefore, the classification of entanglement via the rank of the coefficient matrix has the significant advantage of being independent of the dimension of state and permutation of qudits. Let $\mathcal{F}_{n,r}$ represents the family of all n-qudit states with rank r. It is clear that all full separable states belong to $\mathcal{F}_{n,1}$. With the help of permutation of qudits, the families $\mathcal{F}_{n,r}$ can be further divided into subfamilies. Define $\mathcal{F}^\sigma_r$ (here we have omitted the subscript n) as the subfamily whose coefficient matrix rank is r with respect to permutation σ. The general expression of the subfamilies is

$$\mathcal{F}^{\sigma_1,\sigma_2,\cdots,\sigma_m}_{r_1,r_2,\cdots r_m} = \mathcal{F}^{\sigma_1}_{r_1} \cap \cdots \cap \mathcal{F}^{\sigma_m}_{r_m}. \tag{10}$$

In order to maximize the number of families, the value of l is given by

$$l = \mathrm{argmax}\{\mathcal{P}(l)\}, \tag{11}$$

where

$$\mathcal{P}(l) = \prod_{\{\sigma\}} \min\{\prod_{k=1}^{l} d_{q_k}, \prod_{k=l+1}^{n} d_{q_k}\} \tag{12}$$

with d_{q_k} the dimension of the party corresponding to q_k. It is obvious that for states with each party of the same dimension, the family number is maximized when $l = [n/2]$.

We now consider the n-qudit GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{d}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n} + \cdots + |d-1\rangle^{\otimes n}). \tag{13}$$

A simple calculation shows that $rank(|GHZ\rangle) = d$.

As an application of the generalized method, consider the following state:

$$\begin{aligned} &|l_1, l_2, n\rangle = \\ &\left(\frac{n!}{l_1! l_2!}\right)^{-\frac{1}{2}} \sum_k P_k \left|\underbrace{1,\cdots,1}_{l_1},\underbrace{2,\cdots,2}_{l_2},\underbrace{0,\cdots,0}_{l_0}\right\rangle, \end{aligned} \tag{14}$$

where $|1\rangle, |2\rangle$ are the excitations, $|0\rangle$ represents the ground state, and l_0, l_1, l_2 are the number of states $|0\rangle, |1\rangle, |2\rangle$, respectively, which satisfy $l_1 + l_2 \le n-1$. $\{P_k\}$ is the set that contains all permutations. We donate the states in Eq. (14) as D^n_3 states.

For D^n_3 states, states $|l_1, l_2, n\rangle$, $|l_2, l_1, n\rangle$, $|n-l_1-l_2, l_1, n\rangle$, $|n-l_1-l_2, l_2, n\rangle$, $|l_1, n-l_1-l_2, n\rangle$, and $|l_2, n-l_1-l_2, n\rangle$ can be transformed into each other under SLOCC, namely, they

belong to the same family. In the following, we can arrange these states and denote them as $a(l_1, l_2, l_0)$, where $l_0 = n - l_1 - l_2$. We study the classification of entanglement of D_3^9 states with respect to l_1, l_2 and l_0. The variance of l_1, l_2 and l_0 and the ranks of the coefficient matrices under different arrangements are shown in Fig. 1, which shows that the rank of the coefficient matrix increases with the decrease of the variance, and most of the D_3^9 states can be distinguished by the ranks of the coefficient matrices.

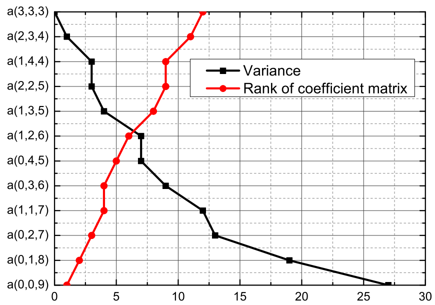

Figure 1: (Color online) Variance of l_1, l_2 and l_0 and ranks of the coefficient matrices under different arrangements (shown in the vertical axis) existing in D_3^9 states.

We then consider D_4^n states. which are defined as

$$|l_1, l_2, l_3, n\rangle = \left(\frac{n!}{l_1! l_2! l_3!}\right)^{-\frac{1}{2}} \sum_k P_k \left| \underbrace{1, \cdots, 1}_{l_1}, \underbrace{2, \cdots, 2}_{l_2}, \underbrace{3, \cdots, 3}_{l_3}, \underbrace{0, \cdots, 0}_{l_0} \right\rangle, \tag{15}$$

where $|1\rangle, |2\rangle$ and $|3\rangle$ are the excitations with l_1, l_2 and l_3 as their numbers, which satisfy $l_1 + l_2 + l_3 \leq n - 1$, whereas $|0\rangle$ is the ground state.

We study the classification of entanglement of D_4^8 states with respect to l_1, l_2, l_3, and l_0. The variance of l_1, l_2, l_3 and l_0 and the ranks of the coefficient matrices under different arrangements are shown in Fig. 2. The rank of the coefficient matrices shows a contrasting trend with the decrease of the variance, and we can distinguish most states in terms of the ranks of the coefficient matrices.

In the end, we discuss the entanglement classification of the $2 \otimes 2 \otimes 2 \otimes 4$ system. For the cases where $l = 1$, $l = 2$, and $l = 3$, the values of $\mathcal{P}(l)$ are 4, 64 and 4, respectively. To maximize the family number, we consider the case where $l = 2$. The set of permutation consists of three elements: $\{\sigma\} = \{\sigma_0 = I, \sigma_1 = (1,3), \sigma_2 = (1,4)\}$. The classification results are shown in Table II. It needs to be noted that the entangled states ($|W\rangle$ and $|GHZ\rangle$ states) in $\mathcal{F}_{2,2,2}^{\sigma_0,\sigma_1,\sigma_2}$ have a similar Frobenius algebra structure [33].

In summary, the rank invariance of the coefficient matrix under SLOCC has been proven to be valid in the n-qudit pure

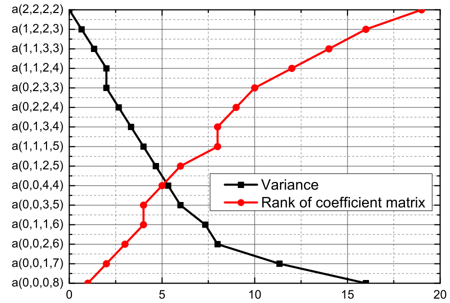

Figure 2: (Color online) Variance of l_1, l_2, l_3 and l_0 and ranks of coefficient matrices under different arrangements (shown in the vertical axis) existing in D_4^8 states.

Table I: SLOCC classification of the $2 \times 2 \times 2 \times 4$ system. The permutations are $\sigma_0 = I, \sigma_1 = (1,3), \sigma_2 = (1,4)$.

SLOCC family	Representative entangled states
$\mathcal{F}_{4,4,4}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 0010\rangle + \lvert 0101\rangle + \lvert 0111\rangle$ $+ \lvert 1002\rangle + \lvert 1012\rangle + \lvert 1103\rangle + \lvert 1113\rangle$
$\mathcal{F}_{4,4,3}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 1010\rangle + \lvert 1001\rangle + \lvert 0102\rangle + \lvert 1113\rangle$
$\mathcal{F}_{4,3,4}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 0110\rangle + \lvert 1100\rangle + \lvert 1002\rangle + \lvert 1113\rangle$
$\mathcal{F}_{3,4,4}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 0110\rangle + \lvert 1100\rangle + \lvert 0012\rangle + \lvert 1113\rangle$
$\mathcal{F}_{4,3,3}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 0111\rangle + \lvert 1012\rangle + \lvert 1113\rangle$
$\mathcal{F}_{3,4,3}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 1101\rangle + \lvert 1012\rangle + \lvert 1113\rangle$
$\mathcal{F}_{3,3,4}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 0111\rangle + \lvert 1102\rangle + \lvert 1113\rangle$
$\mathcal{F}_{4,4,2}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 1010\rangle + \lvert 0102\rangle + \lvert 1113\rangle$
$\mathcal{F}_{4,2,4}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 0110\rangle + \lvert 1002\rangle + \lvert 1113\rangle$
$\mathcal{F}_{2,4,4}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 1100\rangle + \lvert 0012\rangle + \lvert 1113\rangle$
$\mathcal{F}_{3,3,3}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 1010\rangle + \lvert 1001\rangle + \lvert 1113\rangle$
$\mathcal{F}_{3,3,2}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 1010\rangle + \lvert 1112\rangle$
$\mathcal{F}_{3,2,3}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 1001\rangle + \lvert 1112\rangle$
$\mathcal{F}_{2,3,3}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle + \lvert 1100\rangle + \lvert 1112\rangle$
$\mathcal{F}_{2,2,2}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 1010\rangle + \lvert 1100\rangle + \lvert 1001\rangle$ $\lvert 0001\rangle + \lvert 0010\rangle + \lvert 0100\rangle + \lvert 1000\rangle$ $\lvert 0000\rangle + \lvert 1111\rangle$
$\mathcal{F}_{2,2,1}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 1100\rangle + \lvert 1001\rangle$
$\mathcal{F}_{2,1,2}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 1100\rangle + \lvert 1010\rangle$
$\mathcal{F}_{1,2,2}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 1010\rangle + \lvert 1001\rangle$
$\mathcal{F}_{1,1,1}^{\sigma_0,\sigma_1,\sigma_2}$	$\lvert 0000\rangle$

states regardless of the dimension of each partite and the permutation of qudits. Numerical results showed that this generalization can investigate the entanglement feature of quantum states with qudits. We have discussed the entanglement classification of the $2 \otimes 2 \otimes 2 \otimes 4$ system and found 19 different SLOCC families with respect to the generalized method.

From the presented examples, the rank of coefficient ma-

trix and the degree of entanglement can be speculated to have some kind of relation (i.e., the states discussed with maximum rank are always maximally entangled). We expect that our generalization could come up with further theoretical and experimental results.

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APPENDIX

Now we prove the following theorem:

Let $|\psi\rangle$, $|\phi\rangle$ be any states in the n-partite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where $\mathcal{H}_i$ is of dimension d_i, $1 \leq i \leq n$. If there exist $A_i \in \mathcal{M}_{d_i}(\mathbb{C})$ $(1 \leq i \leq n)$ such that

$$|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\phi\rangle, \tag{16}$$

then, for any $1 \leq l < n$,

$$M(|\psi\rangle) = A_1 \otimes \cdots \otimes A_l M(|\phi\rangle)(A_{l+1} \otimes \cdots \otimes A_n)^T. \tag{17}$$

We will prove Eq. (17) by the induction method. Clearly, if $A_i = I_i$ (the identity matrix in $\mathcal{M}_{d_i}(\mathbb{C})$) for every $1 \leq i \leq n$, then equation Eq. (8) holds.

Let $|\psi\rangle = \sum_{i=0}^{d_1 \cdots d_n - 1} c_i |i\rangle$ and for $1 \leq r < n$,

$$|\psi\rangle = I_1 \otimes \cdots \otimes I_r \otimes A_{r+1} \otimes \cdots \otimes A_n |\phi\rangle. \tag{18}$$

For any $1 \leq l < n$, we assume that

$$\begin{aligned} M(|\psi\rangle) &= I_1 \otimes \cdots \otimes I_r \otimes A_{r+1} \\ &\quad \otimes \cdots \otimes A_l M(|\phi\rangle)(A_{l+1} \otimes \cdots \otimes A_n)^T, \\ \text{when} &\quad r+1 \leq l < n; \\ M(|\psi\rangle) &= I_1 \otimes \cdots \otimes I_l M(|\phi\rangle) \\ &\quad \times (I_{l+1} \otimes \cdots \otimes I_r \otimes A_{r+1} \otimes \cdots \otimes A_n)^T, \\ \text{when} &\quad 1 \leq l < r < n; \\ M(|\psi\rangle) &= I_1 \otimes \cdots \otimes I_l M(|\phi\rangle)(A_{r+1} \otimes \cdots \otimes A_n)^T, \\ \text{when} &\quad 1 \leq l = r < n. \end{aligned} \tag{19}$$

Next, we will prove that when

$$|\psi'\rangle = I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes \cdots \otimes A_n |\phi\rangle, \tag{20}$$

there is

$$\begin{aligned} M(|\psi'\rangle) &= I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes \cdots \otimes A_l \\ &\quad \times M(|\phi\rangle)(A_{l+1} \otimes \cdots \otimes A_n)^T, \\ \text{when} &\quad r+1 \leq l < n; \\ M(|\psi'\rangle) &= I_1 \otimes \cdots \otimes I_l M(|\phi\rangle) \\ &\quad \times (I_{l+1} \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes \cdots \otimes A_n)^T, \\ \text{when} &\quad 1 \leq l < r < n; \\ M(|\psi'\rangle) &= I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r M(|\phi\rangle)(A_{r+1} \otimes \cdots \otimes A_n)^T, \\ \text{when} &\quad 1 \leq l = r < n. \end{aligned} \tag{21}$$

Write $|\psi'\rangle = \sum_{i=0}^{d_1 \cdots d_n - 1} b_i |i\rangle$ and

$$A_r = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d_r} \\ a_{21} & a_{22} & \cdots & a_{2d_r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{d_r1} & a_{d_r2} & \cdots & a_{d_rd_r} \end{pmatrix}. \tag{22}$$

Since

$$|\psi'\rangle = I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes I_{r+1} \otimes \cdots \otimes I_n |\psi\rangle, \tag{23}$$

we need only prove that

$$\begin{aligned} M(|\psi'\rangle) &= I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes I_{r+1} \otimes \cdots \otimes I_l M(|\psi\rangle), \\ \text{when} &\quad r+1 \leq l < n; \\ M(|\psi'\rangle) &= I_1 \otimes \cdots \otimes I_l M(|\psi\rangle) \\ &\quad \times (I_{l+1} \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes I_{r+1} \otimes \cdots \otimes I_n)^T, \\ \text{when} &\quad 1 \leq l < r; \\ M(|\psi'\rangle) &= I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r M(|\psi\rangle), \\ \text{when} &\quad 1 \leq r = l < n. \end{aligned} \tag{24}$$

From Eq. (23), it can be computed that

$$\begin{aligned} b_{khd_r+s+(t-1)h} &= a_{t1} c_{khd_r+s} + a_{t2} c_{khd_r+h+s} \\ &\quad + \cdots + a_{td_r} c_{khd_r+(d_r-1)h+s}, \end{aligned} \tag{25}$$

where $t = 1, 2, \ldots, d_r$, $k = 0, 1, \ldots, d_1 \cdots d_{r-1} - 1$, $s = 0, 1, \ldots, d-1$, $h = d_{r+1} \cdots d_n$. If $r+1 \leq l < n$, write

$$M(|\psi\rangle) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{d_{l+1} \cdots d_n - 1} \\ c_{d_{l+1} \cdots d_n} & c_{d_{l+1} \cdots d_n + 1} & \cdots & c_{2d_{l+1} \cdots d_n - 1} \\ \vdots & \vdots & \vdots & \vdots \\ c_d & c_{h+1} & \cdots & c_{h+d_{l+1} \cdots d_n - 1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{(d_1 \cdots d_l - 1)d_{l+1} \cdots d_n} & c_{(d_1 \cdots d_l - 1)d_{l+1} \cdots d_n + 1} & \cdots & c_{d_1 \cdots d_n - 1} \end{pmatrix}; \tag{26}$$

if $1 \leq l < r < n$, write

$$M(|\psi\rangle) = \begin{pmatrix} c_0 & c_1 & \cdots & c_h & \cdots & c_{d_{l+1}\cdots d_n - 1} \\ c_{d_{l+1}\cdots d_n} & c_{d_{l+1}\cdots d_n + 1} & \cdots & c_{d_{l+1}\cdots d_n + d_{r+1}\cdots d_n} & \cdots & c_{2d_{l+1}\cdots d_n - 1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{(d_1\cdots d_l - 1)d_{l+1}\cdots d_n} & c_{(d_1\cdots d_l - 1)d_{l+1}\cdots d_n + 1} & \cdots & c_{(d_1\cdots d_l - 1)d_{l+1}\cdots d_n + h} & \cdots & c_{d_1\cdots d_n - 1} \end{pmatrix}; \quad (27)$$

if $1 \leq l = r < n$, write

$$M(|\psi\rangle) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{h-1} \\ c_d & c_{d+1} & \cdots & c_{2h-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{(d_1\cdots d_l - 1)h} & c_{(d_1\cdots d_l - 1)h+1} & \cdots & c_{d_1\cdots d_n - 1} \end{pmatrix}, \quad (28)$$

then it follows from Eq. (25) that equations Eq. (24) holds.

Finally, we consider the permutation of qudits. Applying the permutation σ defined in Eq. (6) to both sides of Eq. (17) and we have

$$\begin{aligned} M^\sigma(|\psi\rangle) &= A_1^\sigma \otimes \cdots \otimes A_l^\sigma \\ &\times M^\sigma(|\phi\rangle)(A_{l+1}^\sigma \otimes \cdots \otimes A_n^\sigma)^T. \end{aligned} \quad (29)$$

When $A_1, \cdots, A_n$ are ILOs, it can be directly concluded from Eq. (29) that $M^\sigma(|\psi\rangle)$ and $M^\sigma(|\phi\rangle)$ have the same rank. Thus two SLOCC equivalent states have the same rank with respect to every permutation of qudits.

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